One World Octagon (P414). David Seppala-Holtzman (St. Joseph's College) sent this problem about One World Trade Center (OWTC) in Manhattan. The building rests on a square prism, 200 feet on a side and 185 feet high. The 150 foot-by-150 foot square parapet is centered over the base viewed from above, rotated 45° relative to the base, and is 1,368 feet high. Each corner of the parapet is connected by straight beams to the two nearest corners of the prism base. Is there a height at which the horizontal cross section of OWTC is a regular octagon? If so, what is it?

Yes, OWTC has a regular-octagonal cross section **861** feet above the ground. We received a solution from the Missouri State University problem-solving group, and partial solutions from Randy K. Schwartz and from problem-solving groups at Cal Poly Pomona, Georgia Southern University, and Skidmore College. Figure 5 helps to visualize the solution.

First, the symmetries of the building around its central vertical axis imply that all cross sections of the building are equiangular. We now follow the method from Missouri State. Consider the cross section at a height that is a fraction $0 < \lambda < 1$

of the way from the top of the prism base to the height of the parapet. Because each of the nearly vertical faces of the tower is an isosceles triangle, by similar triangles half of the edges of the cross section have length 150λ and the other half have length $200(1-\lambda)$. In the regular cross section, these are all equal, yielding $\lambda = 4/7$. This value means that the regular cross section occurs at a height of $185 + (4/7)(1368 - 185) = 861$ feet.

Small Squiral (P415). Let square S_0 have vertices $A_0 = (0,1), B_0 = (-1,0), C_0 = (0,-1)$, and $D_0 = (1,0)$. A *squiral* is a finite sequence of n squares $S_i = (A_i, B_i, C_i, D_i)$ with A_{i+1} on side A_iB_i, B_{i+1} on B_iC_i, and C_{i+1} on C_iD_i, and such that each line B_iA_i has positive slope, except B_nA_n is vertical.

The *size* of a squiral is the side length of the final square $\overline{A_nB_n}$. What is the smallest possible size of any squiral?

The infimum of squiral sizes (not achieved) has the lovely expression $\sqrt{2}e^{-\tau/8}$ (where $\tau = 2\pi$ is the angle measure of a circle). We received insightful submissions from Lucas and Alex Perry (San Francisco University HS) and Randy K. Schwartz, and, for the first time for a Monkey Bars problem, complete solutions from Jesús Sistos together with Jesús Liceaga and the Eagle Problem Solvers (Georgia State University), from the CNU Problem Solving Seminar, and from the Missouri State University problem-solving group.

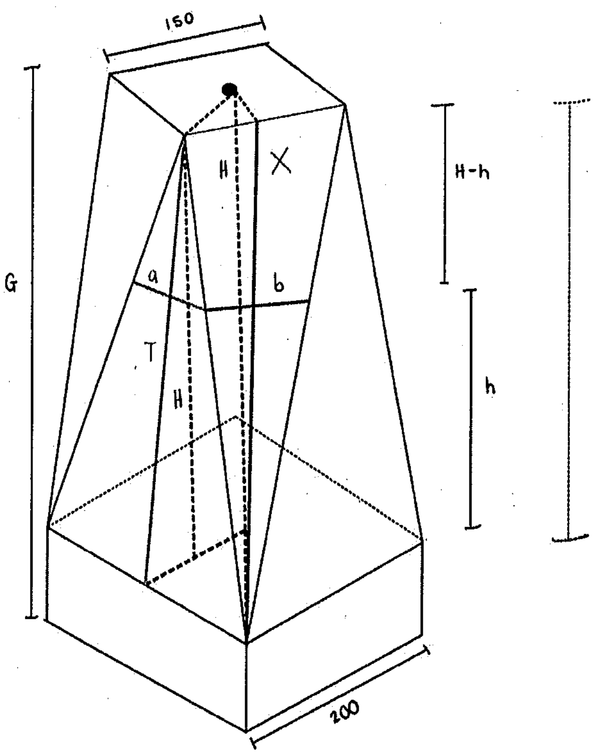

Figure 5. A geometric schematic of One World Trade Center provided by the Cal Poly Pomona group.

All submissions identified the key angle θ as shown in figure 6 (provided by the Perry brothers).

Because s_2, the side of the inner square, is the hypotenuse of a right triangle with angle θ at vertex E, some trigonometry tells us that

$$\frac{s_2}{s_1} = \frac{1}{\sin\theta + \cos\theta}.$$

Thus, to find the smallest squiral, we seek a sequence of angles $\theta_1, \ldots, \theta_n$ that sums to $\alpha = \tau/8$ (the squares overall rotate an eighth of a turn) and that maximizes the product

$$\prod_{i=1}^{n} \sin\theta_i + \cos\theta_i.$$

For a fixed n, this maximum occurs when each θ_i is equal to α/n. Sistos proved this maximum by combining the arithmetic-geometric mean and Jensen's inequalities; the CNU Problem Seminar used Lagrange multipliers; and the Missouri State group found a contradiction presuming the maximum occurred with any θ_i and θ_j distinct.

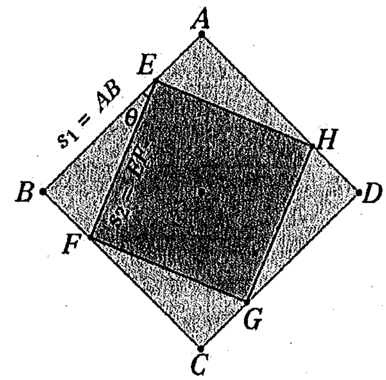

Figure 6. Two generic consecutive squiral squares.

All groups noted that $(\sin(\alpha/n) + \cos(\alpha/n))^n$ is a strictly increasing sequence. Hence, there is no minimum squiral size, but the infimum is $\sqrt{2}$ (the initial square side) divided by the limit of this sequence. It's easier to find the logarithmic limit:

$$\alpha \cdot \lim_{n\to\infty} \frac{\ln(\sin(\alpha/n) + \cos(\alpha/n))}{\alpha/n} = \alpha \cdot \lim_{x\to 0} \frac{\ln(\sin x + \cos x)}{x}.$$

Using L'Hôpital's rule, we see the limit is $\alpha \cdot 1$, for a minimum squiral size of $\sqrt{2}/e^{\alpha} = \sqrt{2}/e^{\tau/8}$.

The Missouri State solvers generalized to a nested spiral of regular k-gons, for which the infimum size has the elegant form $2\sin(\alpha)e^{-\alpha\tan\alpha}$ where now $\alpha = \tau/2k$.